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agitated for a century was satisfactorily treated within the compass of less than a couple of dozen lines.

Did Euler communicate his results on logarithms to his old and revered master, John Bernoulli I? Historians have given no information on this question. But there is evidence that Euler did so. Pessuti, who took part in the controversy in Italy, states that he met Euler in St. Petersburg and that Euler made to him the statement that "having communicated his [Euler's] results to John Bernoulli shortly before his death, the good old man replied that he died content, because he saw reconciled what seemed to be irreconcilable paradoxes, namely the contradictions which had given rise to the great dispute on logarithms of negative numbers, long continued, between Leibniz and himself."[1] Calandrelli replies to Pessuti that this statement cannot be true, since Bernoulli died in 1748, and Euler's article was sent to the Berlin Academy in 1749. But we know now that Euler had his first paper completed in 1747.

(*To be continued.*)

AMICABLE NUMBER TRIPLES.[2]

By L. E. DICKSON, University of Chicago.

1. Two numbers are called amicable if each is the sum of the proper divisors of the other, a proper divisor of n being a divisor less than n. We shall say that three numbers form an amicable triple if the sum of the proper divisors of each equals the sum of the remaining two numbers. Let $\sigma(n)$ denote the sum of all the divisors of n. Then n_1, n_2, n_3 form an amicable triple if

$$\sigma(n_1) = \sigma(n_2) = \sigma(n_3) = n_1 + n_2 + n_3. \tag{1}$$

Similarly, $n_1, \cdots, n_k$ form an amicable k-triple if

$$\sigma(n_1) = \sigma(n_2) = \cdots = \sigma(n_k) = n_1 + n_2 + \cdots + n_k. \tag{2}$$

If $k = 2$, n_1 and n_2 are amicable numbers in the usual sense.

For $k > 2$, I have verified that the only solutions of (2) in which not every n_i exceeds 1,000 are those with $k = 3$, $n_1 = n_2 = n_3 = 120$ or 672. When the k numbers n_i are all equal, n_1 is called a multiply perfect number of multiplicity k; as many as 251 such numbers are known,[3] all with $k \leqq 7$.

I obtain below 8 sets of amicable triples in which two of the numbers are equal, and the triple of distinct numbers (end of § 8):

$$293 \cdot 337a, \quad 5 \cdot 16561a, \quad 99371a, \quad a = 2^5 \cdot 3 \cdot 13. \tag{I}$$

Another amicable triple of distinct numbers is obtained in § 14 by a device.

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[1] G. Calandrelli, *Saggio analitico, etc.*, Roma, 1778, p. 14.

[2] Read before the American Mathematical Society, December 31, 1912.

[3] Cf. Carmichael and Mason, *Proceedings Indiana Academy of Science*, 1911, pp. 257–270.

2. Consider an amicable triple apq, ars, at, where p, q, r, s, t are primes not dividing a, and $p \neq q$, $r \neq s$. Then

$$(p+1)(q+1) = (r+1)(s+1) = t+1.$$

Call x the greatest common divisor of $p+1$, $r+1$, and set $p+1 = \alpha x$, $r+1 = \beta x$. Then α, β are relatively prime, and

$$q+1 = \beta y, \quad s+1 = \alpha y, \quad t+1 = \alpha\beta xy,$$

$$\alpha\beta xy\sigma(a) = \sigma(at) = a(pq+rs+t) = a(3\alpha\beta xy - \alpha x - \alpha y - \beta x - \beta y + 1).$$

We therefore set

$$\frac{a}{3a-\sigma(a)} = \frac{b}{c} \quad (b, c \text{ relatively prime}). \tag{3}$$

Hence

$$c\alpha\beta xy - b(\alpha+\beta)(x+y) + b = 0, \tag{4}$$

$$[c\alpha\beta x - b(\alpha+\beta)][c\alpha\beta y - b(\alpha+\beta)] = b^2(\alpha+\beta)^2 - bc\alpha\beta. \tag{5}$$

We assign values to a, α, β, determine b and c by (3), express in all possible ways the second member of (5) as a product of two integers, and take the latter as the values of the two factors on the left of (5). A favorable case is that in which $c\alpha\beta$ is small.

For a even, $p, \cdots, t$ are odd, x and y both even, so that, by (4), b is a multiple of 4. First, let α be even and β odd. Then b is a multiple of 8. For $a = 96$, $b = 8$, $c = 3$, and $\alpha/2$ is odd by (4); for $\alpha = 2$ or 6, $\beta \leqq 19$; $\alpha = 10$ or 14, $\beta \leqq 10$, (5) has no integral solutions except those making $p = 3$, a divisor of a. Next, let α and β be odd. For $a = 96$; $a = 40$ or 224, $b = 4$, $c = 3$; $a = 48$, $b = 12$, $c = 5$, the small values of α, β were examined, but the only solution found is that with $a = 96$, $\alpha = \beta = 1$, viz.,

$$2^5 \cdot 3 \cdot 5 \cdot 43, \quad 2^5 \cdot 3 \cdot 5 \cdot 43, \quad 2^5 \cdot 3 \cdot 263. \tag{II}$$

While this method is less effective than the following one, it introduces naturally the substitution (3), used also below.

3. Let apq, arf, at form an amicable triple, where p, q, r, t are primes not dividing a, and f is an integer relatively prime to a. Call h the greatest common divisor of $\sigma(f)$ and $p+1$, and set $\sigma(f) = gh$, $p+1 = xh$, so that g and x are relatively prime. Then, by (1),

$$r+1 = xy, \quad q+1 = gy, \quad t = ghxy - 1,$$

$$0 = xy[2agh + af - gh\sigma(a)] - a(hx + gy + f).$$

Multiply by b/a and eliminate $\sigma(a)$ by use of (3). We get

$$exy - bhx - bgy - bf = 0, \quad e \equiv bf - bgh + cgh. \tag{6}$$

Multiply (6_1) by egh. We get $MN = T$, where

$$M = h(ex - bg), \quad N = g(ey - bh), \quad T = (ef + bgh)bgh.$$

Solving for x, y, we find that

$$(7) \quad p = \frac{M + b\sigma(f)}{e} - 1, \quad q = \frac{N + b\sigma(f)}{e} - 1, \quad r = \frac{(p+1)(q+1)}{\sigma(f)} - 1,$$

$$(8) \qquad e = bf - b\sigma(f) + c\sigma(f), \quad T = [ef + b\sigma(f)]b\sigma(f).$$

Note that g, h do not occur explicitly in (7), (8). The problem is therefore to assign such relatively prime values to a and f that, when T is expressed as a product MN, the numbers (7) will be integers and primes and

$$(9) \qquad t = (p+1)(q+1) - 1$$

a prime, no one of the four primes to divide a.

With the exception of 945, every odd number $a < 1000$ is deficient, viz., $\sigma(a) < 2a$, and gives $c > b$, whence e is a sum of positive multiples of f and $\sigma(f)$. Thus the denominator e is large compared with f.

Henceforth, we shall assume that a is even. Then p, q, r, f, t are odd, and xy, hx, gy are even. Thus, by (6), b is even. If x, y are not both even, h or g is even and e is even. Hence in every case, exy and therefore also b is divisible by 4. Then by (3), a is either a multiple of 8 or is the product of 4 and an odd square.

4. We have shown that, when a is even, $b = 2^\beta B$, where $\beta \geqq 2$ and B is odd. We proceed to prove the noteworthy and very useful

THEOREM: *There is no amicable triple with $\sigma(f)$ divisible by 2^β.*

Suppose that $\sigma(f) = 2^\beta \phi$. Write γ for the odd number $b - c$. Then by (8)

$$(10) \qquad e = 2^\beta k, \quad k \equiv Bf - \gamma\phi,$$

$$(11) \qquad MN = 2^\beta A \cdot 2^\beta B\sigma(f), \quad A \equiv kf + B\sigma(f).$$

First, let ϕ be even. Then k and A are odd. By (7), M and N are multiples of $2 \cdot 2^\beta$, since p and q are odd. Thus

$$M = 2^{\beta+1}m, \quad N = 2^{\beta+1}n, \quad 4mn = AB\sigma(f),$$

$$k(p+1) = 2m + B\sigma(f), \quad k(q+1) = 2n + B\sigma(f),$$

$$k^2(r+1) = \frac{4mn}{\sigma(f)} + E = AB + E, \quad E \equiv 2B(m+n) + B^2\sigma(f).$$

Since E is even, $r + 1$ is odd, whereas r is to be an odd prime.

Next, let ϕ be odd. Then $k = 2^\kappa K$, $\kappa \geqq 1$, K odd. By (7),

$$(12) \qquad 2^{\beta+\kappa}K(p+1) = M + 2^{2\beta}B\phi.$$

Consider the sub-case $\kappa \geqq \beta - 1$. Since p is odd, the left member of (12) is divisible by $2^{2\beta}$. The same is true of M and of N. Set

$$M = 2^{2\beta}m, \quad N = 2^{2\beta}n.$$

By (11), A and therefore also k is divisible by 2^{β}. Thus

$$\kappa \geqq \beta, \quad A = 2^{\beta}\alpha, \quad \alpha \equiv 2^{\kappa-\beta}fK + B\phi.$$

By (12),

$$0 \equiv 2^{2\beta}(m + B\phi)(\text{mod } 2^{\beta+\kappa+1}), \quad m \equiv - B\phi(\text{mod } 2^{\kappa+1-\beta}).$$

From the latter and the like congruence for n, we get

$$mn \equiv B^2\phi^2(\text{mod } 2^{\kappa+1-\beta}).$$

But, by (11), $mn = \alpha B\phi$. Replacing α by its value, we get

$$2^{\kappa-\beta}fKB\phi + B^2\phi^2 = mn \equiv B^2\phi^2(\text{mod } 2^{\kappa+1-\beta}),$$

which is impossible since $fKB\phi$ is odd.

Finally, let $\kappa < \beta - 1$. By (12), M is divisible by $2 \cdot 2^{\beta+\kappa}$. Let

$$M = 2^{\beta+\kappa+1}m, \quad N = 2^{\beta+\kappa+1}n, \quad \alpha = fK + 2^{\beta-\kappa}B\phi.$$

Then α is odd. By (11),

$$A = 2^{\kappa}\alpha, \quad mn = 2^{\beta-\kappa-2}\alpha B\phi.$$

By (12) or (7),

$$K(p + 1) = 2m + 2^{\beta-\kappa}B\phi, \quad K(q + 1) = 2n + 2^{\beta-\kappa}B\phi,$$

$$K^2(r + 1) = \frac{4}{2^{\beta}\phi}(m + 2^{\beta-\kappa-1}B\phi)(n + \cdots) = \frac{1}{2^{\kappa}}[\alpha B + 2B(m + n) + 2^{\beta-\kappa}B^2\phi].$$

The number in brackets is odd. Hence $r + 1$ is not an integer.

For the case f a prime, the theorem yields the

COROLLARY: There is no amicable triple apq, arf, at, where p, q, r, f, t are primes not dividing the even number a, and at least one of the primes p, q, r, f is of the form $2^{\beta}\phi - 1$, where 2^{β} is the highest power of 2 dividing b, defined by (3).

5. Applying the theorem for $\beta = 2$ or 3, and finding the solutions f of $\sigma(f) = w, 2w, 4w$, where w is odd, we obtain the

THEOREM. *If $b = 4B$, B odd, an amicable triple has*

$$f = (4k + 1)^{4l+1}g^2,$$

where $4k + 1$ is unity or a prime not dividing g. If $b = 8B$, f may have also one of the following values

$$(8k + 3)^{4l+1}g^2, \quad (4k + 1)^{8l+3}g^2, \quad (4k + 1)^{4l+1}(4j + 1)^{4v+1}g^2,$$

where $8k + 3$, $4k + 1$, $4j + 1$ are primes not dividing g.

6. The case $c = 1$ is favorable since e is then small in comparison with f. The minimum b is then 4. The condition for $b = 4$, $c = 1$ is, by (3), $11a = 4\sigma(a)$.

The least k for which $\sigma(2^k)$ has the factor 11 is $k = 9$. Let $a = 2^9\alpha$, α odd. Then $2^7\alpha = 3 \cdot 31\sigma(\alpha)$. Set $\alpha = 31\gamma$. Then $3\sigma(\gamma) = 4\gamma$, which holds for $\gamma = 3$. Thus a solution is $a = 2^9 \cdot 3 \cdot 31$. For $f = 1$, we obtain the solution

$$\text{(III)} \qquad 5 \cdot 13a, \quad 83a, \quad 83a, \quad a = 2^9 \cdot 3 \cdot 31.$$

Next, let f be a prime. By § 5, $f = 4l + 1$. Then $e = 2(2l - 1)$. Thus

$$M = 4m, \quad mn = (2l + 1)(8l^2 + 6l + 3), \quad (2l - 1)\ (p + 1) = 2(m + 4l + 2).$$

Hence m and n are $\equiv - 4(\text{mod } 2l - 1)$. For $f = 5$, $m = 3$, $n = 17$, $q = 45$. For $f = 13$, $m = 1$, $n = 3 \cdot 7 \cdot 31$, $q = 265$, or $m = 21$, $n = 31$, which yields the solution

$$\text{(IV)} \qquad 13 \cdot 17a, \quad 13 \cdot 17a, \quad 251a, \quad a = 2^9 \cdot 3 \cdot 31.$$

For $f = 37$, $m = 3 \cdot 5 \cdot 19$, $n = 47$, $q = 9$. For $f = 137$, $m = 3 \cdot 5 \cdot 31$, $n = 23 \cdot 61$, $q = 45$. For the remaining primes $4l + 1 < 200$, there is no set of values m, n; likewise, not for $f = 7^2, 11^2, 17^2$. For $f = 5^2$, $e = 7$, $M = 2m$, $m \equiv 1(\text{mod } 7)$, $m = 1$, $n = 13 \cdot 23 \cdot 31$, $q = 2665$. For $f = 13^2$, $e = 127$, $m \equiv 15(\text{mod } 127)$, $m = 15$, $n = 23 \cdot 61 \cdot 193$, $q \equiv 0(\text{mod } 3)$. The smallest $f = v^2w^2$ (v, w distinct primes) making $e > 0$ are $5^2 \cdot 17^2$ and $7^2 \cdot 11^2$, giving $e = 349$ and 973, respectively. Each prime factor of f exceeds 3 by the

THEOREM. *Each prime factor of f exceeds $b/c - 1$.*

Proof is needed only when $b > c$. If v is a prime factor of f, we have, since $e > 0$,

$$\sigma(f) \geqq f + \frac{f}{v}, \quad bf > (b - c)\left(f + \frac{f}{v}\right), \quad c > \frac{(b - c)}{v}.$$

7. Let $b = 12$, $c = 1$. The condition (3) is $12\sigma(a) = 35a$. For $a = 2^3\alpha$, α odd, $3^2\sigma(\alpha) = 2 \cdot 7\alpha$; let $\alpha = 3^2\gamma$; then $13\sigma(\gamma) = 14\gamma$, which holds for $\gamma = 13$. For $a = 2^5\alpha$, $3^3\sigma(\alpha) = 2^3 \cdot 5\alpha$, which holds for $\alpha = 3^3$.

For $f = 1$, we have $e = 1$, $M = 2m$, $mn = 3 \cdot 13$, $m = 1$ or 3,

$$(p, q, r) = (13, 89, 1259) \text{ or } (17, 37, 683),$$

all primes, and $t = r$. We thus have the three amicable triples

$$\text{(V)} \qquad 13 \cdot 89a, \quad 1259a, \quad 1259a, \quad a = 2^5 \cdot 3^3,$$

$$\text{(VI), (VII)} \quad 17 \cdot 37a, \quad 683a, \quad 683a, \quad a = 2^5 \cdot 3^3 \text{ or } 2^3 \cdot 3^2 \cdot 13.$$

Let f be a prime (> 11, by § 6). Then $f = 4k + 1$, $M = 4m$, $N = 4n$, $e = 4k - 10$, $mn = 3(2k + 1)(8k^2 + 6k + 7)$, $(2k - 5)(p + 1) = 2m + 12(2k+1)$. Thus $m \equiv - 36(\text{mod } 2k - 5)$. If $k = 9j + 7$, $2k - 5 = 9(2j + 1)$, $m = 9m'$, $n = 9n'$, $m'n' = (6j + 5)(72j^2 + 118j + 49)$, $(2j + 1)(p + 1) = 2m' + 4(6j + 5)$, $m' \equiv - 4(\text{mod } 2j + 1)$. For $j = 0$, $f = 29$, $m'n' = 5 \cdot 7^2$, while $m' = 1$, 7 or 7^2 gives $p \equiv 0(\text{mod } 3)$. For $j = 1$, f is composite. For $j = 2$, $f = 101$, $m'n' = 3 \cdot 17 \cdot 191$; either $m' = 1$, $q = 3 \cdot 1303$, or $m' = 51$, $p = 3 \cdot 11$.

For $j = 3$, $f = 137$, there is no set m, n. For $j = 4$, $f = 173$, $m' = 7 \cdot 29$, $n' = 239$, $p = 3 \cdot 19$. Next, if $k = 3v + 1$, $v \not\equiv 3j + 2$, $m = 3m'$ and

$$m'n' = 3(2v + 1)(24v^2 + 22v + 7), \quad (2v - 1)(p + 1) = 2m' + 12(2v + 1),$$
$$m' \equiv -12 (\mathrm{mod}\ 2v - 1).$$

For $v = 1, f = 17$, $m'n' = 9 \cdot 53$; if $n' = 53$ or $9 \cdot 53$, $q = 3 \cdot 47$ or $23 \cdot 43$; if $n' = 3 \cdot 53, p = 41, q = 353$, but $r = 825$. For $v = 3, f = 41, m'n' = 3 \cdot 7 \cdot 17^2$, $m' = 3$, $p = 17$, $q = 825$. For $v = 4, f = 53$, $m'n' = 3^3 \cdot 479$, $m' = 9$, $p = 17$, $q = 425$. For $v = 6$, 10, 13, 15, f is composite. For $v = 7$, $f = 89$, $m'n' = 3^2 \cdot 5 \cdot 7 \cdot 191$; either $m' = 1$, $p = 13$, $q = 13 \cdot 23 \cdot 31$ or $m' = 3 \cdot 5 \cdot 7$, $p = 29$, $q = 101$, $r = 33$. For $v = 9, f = 113$ and for $v = 16, f = 197$, there is no set m, n.

For $k \not\equiv 1 (\mathrm{mod}\ 3)$ the primes $f < 200$ for which there occur sets m, n are $f = 13, 37, 97$. For $f = 13$, $mn = 3 \cdot 7 \cdot 97$; if $m = 1$ or 21, p has the factor 5; if $m = 3$ or 7, q has the factor 11 or 5. For $f = 37$, $m = 3$, $n = 19 \cdot 709$, $p = 17$ and $q = 2089$ are primes, but $r = 23 \cdot 43$. For $f = 97$, $m = 7$, $n = 3 \cdot 7 \cdot 4759$, $p = 13$, $q = 59 \cdot 79$.

Next, let $f = v^2$, v a prime > 11. For $v = 13$, $M = 6m$, $mn = 3 \cdot 19 \cdot 61 \cdot 83$, either $m = 19$, $p = 3 \cdot 51$ or $m = 19 \cdot 61$, $p = 609$. For $v = 19$, $M = 6m$, $mn = 3 \cdot 11 \cdot 41^2 \cdot 127$, $n = 41 \cdot 127$, $q = 11 \cdot 23$. For $v = 17$ or 23, there is no set m, n.

For $f = v^2w^2$, where v and w are distinct primes $\leqq 29$, e is positive only for $v = 23$, $w = 29$, and $e = 40375$.

8. A very favorable case is that in which $b\sigma(f)$ and hence M and N are multiples of e, since the computed values of p, q are always integers. This case does not arise for $f = 1$, whence $\sigma = 1$, $e = c$, since c is prime to b. Let f be a prime. Then $e = c\sigma - b$. Let g be the greatest common divisor of $b = g\beta$ and $\sigma = g\phi$, so that ϕ is prime to β and hence to $E = c\phi - \beta$. Since $e = gE$ divides $gb\phi$, E divides b and therefore also g. Let $g = EQ$. Then

$$b = \beta EQ, \quad \sigma(f) = \phi EQ, \quad e = E^2Q.$$

Conversely, this e divides $b\sigma(f)$.

First, let $f = 5$. Since $\sigma = 6$ and b is a multiple of 4, $g = 2$ or 6, and β is even, E odd. If $g = 6$, $\phi = 1$, $c - \beta = E = 1$ or 3, $b = 6\beta$. In the latter case, $\beta = c - 3$, and c is prime to 6. For $c = 5$, $b = 12$ and we may take $a = 48$, an unfruitful case (§ 2). For $c = 7$, $b = 24$ and we may take $a = 72$. Then for $f = 5$, $e = 18$, $M = 36m$, $mn = 26$, $p + 1 = 2m + 8$; if $m = 1$, $p = 3^2$; if $m = 2, p = 11, q = 3 \cdot 11$. For $f = 11, e = 60, M = 24m, mn = 6 \cdot 79$, $p + 1 = (2m + 24)/5$, $m \equiv 3 (\mathrm{mod}\ 5)$, $m = 3$, $p = 5$, $q = 67$, $r = 3 \cdot 11$. For $f = 13, 17, 19, 25$, there is no set m, n.

Next, let $g = 2$. Then $\phi = 3$, $E = 1$, $b = 2\beta$, $\beta = 3c - 1$. If $c = 1$, $b = 4$, a case treated in § 6. If $c = 3$, $b = 16$ and (3) gives $45a = 16\sigma(a)$. The simplest solution arises from $a = 2^7\alpha$, α odd. Then $17\sigma(\alpha) = 8 \cdot 3\alpha$. Let $\alpha = 17\gamma$, γ prime to 17. Then $3\sigma(\gamma) = 4\gamma$, $\gamma = 3$, whence $a = 2^7 \cdot 3 \cdot 17$.

For $f = 1$, $e = 3$, $M = 2m$, $mn = 4 \cdot 19$, either $m = 1$, $p = 5$, $q = 55$ or $m = 4$, $p = 7$, $q = 17$, $r = 11 \cdot 13$. For $f = 5$, $e = 2$, $M = 4m$, $mn = 2^2 \cdot 3 \cdot 53$, $p + 1 = 2m + 48$, $m \not\equiv 2 (\mathrm{mod}\ 3)$, whence either $m = 1$, $p = 7^2$ or $m = 6$, $p = 59$, $q = 3 \cdot 51$. For $f = 7$, $e = 8$, $M = 16m$, $mn = 4 \cdot 23$, $p + 1 = 2m+16$, whence

m	p	q	r	t
1	17	119	449	$59 \cdot 61$
2	19	107	269	$17 \cdot 127$
4	23	61	$5 \cdot 37$	

For $f = 11$, $e = 20$, $M = 8m$, $mn = 4 \cdot 3 \cdot 103$, either $m = 1$, $p = 9$ or $m = 6$, $p = 11$, $q = 7 \cdot 13$. For $f = 17$, $e = 38$, $M = 4m$, $m = 4$, $n = 3^2 \cdot 467$, $p = 7$, $q = 449$, $r = 199$, $t = 59 \cdot 61$. For $f = 19$, $e = 4 \cdot 11$, $m = 4$, $n = 5 \cdot 17^2$, $p = 7$, $q = 269$, $r = 107$, $t = 17 \cdot 127$. For $f = 23$, $e = 8 \cdot 7$, $M = 16m$, $mn = 4 \cdot 3 \cdot 11 \cdot 19$, either $m = 4$, $p = 7$, $q = 185$, or $m = 11$, $p = 9$. For $f = 29$, $M = 4m$, $m = 5 \cdot 13$, $n = 2^2 \cdot 3 \cdot 101$, $p = 9$. For $f = 13$ or 7^2, there is no set m, n.

If $c = 5$ or 7, the solution of (3) is not evident. Let $c = 9$. Then $b = 4 \cdot 13$, $4 \cdot 13\sigma(a) = 3 \cdot 7^2 \cdot a$. Let $a = 13\alpha$, α not divisible by 13. Then $2^3\sigma(\alpha) = 3 \cdot 7\alpha$. Let $\alpha = 2^5\gamma$. Then $3\sigma(\gamma) = 4\gamma$, $\gamma = 3$, $a = 2^5 \cdot 3 \cdot 13$. For $f = 1$, $e = 9$, $M = 2m$, $m = 1$, $n = 13 \cdot 61$, giving the amicable triple

$$5 \cdot 181a, \quad 1091a, \quad 1091a, \quad a = 2^5 \cdot 3 \cdot 13. \tag{VIII}$$

For $f = 5$, $e = 2$, $M = 4m$, $mn = 3 \cdot 7 \cdot 13 \cdot 23$, $p + 1 = 2(m + 78)$, $m \not\equiv 2$, $n \not\equiv 2 (\mathrm{mod}\ 3)$. We may take n a multiple of $3 \cdot 23$. For $m = 1$, $p = 157$ and $q = 12713$ are primes, but $r = 53 \cdot 6317$. For $m = 7$, $p = 13^2$. For $m = 13$, $p = 181$, $q = 19 \cdot 59$. For $m = 7 \cdot 13$, we obtain solution I of § 1. For $f = 13$ or 17, there is no set m, n.

9. In the intial work of § 8, let $f = 17$. Then $g = 2$, 6 or 18. To make $e = gE$ a minimum, we take $E = 1$. Then for $g = 18$, $\phi = 1$, $b = 18\beta$, $\beta = c-1$, c prime to 18. For $c = 5$, 7 or 11, the solution of (3) is not evident. For $c = 13$, $b = 2^3 \cdot 3^3$, $5 \cdot 127a = 2^3 \cdot 3^3\sigma(a)$. Take $a = 2^6\alpha$, α odd. Then $2^3 \cdot 5\alpha = 3^3\sigma(\alpha)$, $\alpha = 3^3$. For $f = 1$, $e = 13$, $M = 2m$, $m = 9$, $n = 2 \cdot 3 \cdot 229$, $p = 17$ and $q = 227$ are primes, but $r = 11 \cdot 373$. For f a prime, $e = 13f - 203$, $f > 14$. For $f = 17$, $e = 18$, $M = 36m$, $mn = 2 \cdot 3^3 \cdot 233$, $p + 1 = 2m + 216$, $m \not\equiv 2 (\mathrm{mod}\ 3)$, $m \neq 2$, $n \neq 233$. For $m = 1$ or 3, p is composite. For $m = 27$, $q = 31 \cdot 37$. For $m = 6$, 9 or 18, p and q are primes, but r is composite. For $f = 19$, $M = 8m$, either $m = 10$, $p = 99$ or $m = 54$, $n = 5 \cdot 1289$, $q = 3 \cdot 423$. Finally, for $c = 17$, $b = 2^5 \cdot 3^2$, $7 \cdot 11^2a = 2^5 \cdot 3^2\sigma(a)$. Let $a = 2^5\alpha$, α odd. Then $11^2\alpha = 3^4\sigma(\alpha)$, $\alpha = 3^4$. For $f = 1$, $e = 17$, $M = 2m$, $mn = 2^3 \cdot 3^2 \cdot 5 \cdot 61$, $p + 1 = 2(m + 144)/17$, either $m = 9$, $p = 17$, $q = 303$ or $m = 2^2 \cdot 3 \cdot 5$, leading to the amicable triple

$$23 \cdot 59a, \quad 1439a, \quad 1439a, \quad a = 2^5 \cdot 3^4. \tag{IX}$$

For f a prime, $e = 17f - 271$, $f > 14$. If $f = 17$, $e = 18$, $M = 36m$, $mn = 2^3 \cdot 3^2 \cdot 5 \cdot 61$, $p + 1 = 2m + 288$, $m, n \not\equiv 2 \pmod 3$. There are 16 cases; but p is a prime only for $m = 10, 40, 3, 12, 36, 15, 30, 90$. If $m = 10$, $q = 4679$, $r = 3 \cdot 26693$. If $m = 40, 3, 36, 15, 30, 90$, q is composite. If $m = 12$, $q = 3947$, $r = 11 \cdot 6221$. For $f = 19$, $e = 4 \cdot 13$, $M = 8m$, $mn = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 241$, $p + 1 = 2(m + 5)/13 + 110$, $m \equiv 8 \pmod{13}$. There are 4 m's. If $m = 8$ or 60, $p = 3 \cdot 37$ or $7 \cdot 17$. If $m = 21$, $p = 113$, $q = 47 \cdot 97$. If $m = 2^3 \cdot 3 \cdot 5 \cdot 7$, $p = 239$, $q = 13 \cdot 17$. For $f = 23$, $e = 120$, $M = 48m$, $mn = 2^3 \cdot 3^2 \cdot 13 \cdot 31$, $p + 1 = 2(m + 144)/5$, $m \equiv 1 \pmod 5$. There are 6 m's. If $m = 1$, $p = 3 \cdot 19$. If $m = 2^2 \cdot 3 \cdot 13$, $p = 7 \cdot 17$. If $m = 6$, $q = 11 \cdot 181$. If $m = 2^3 \cdot 3^2 \cdot 13$, $q = 3 \cdot 23$. If $m = 2^2 \cdot 3^2$, $p = 71$, $q = 379$, $r = 17 \cdot 67$. If $m = 26$, $p = 67$, $q = 503$, $r = 1427$ are primes, but $t = 43 \cdot 797$.

10. For $a = 144 = b$, $c = 29$, $f = 5$, $e = 30$, $M = 12m$, $mn = 2^2 \cdot 3^2 \cdot 13^2$, $p + 1 = (2m + 144)/5$, $m \equiv 3 \pmod 5$; $m = 2 \cdot 3^2$ or 13 gives $p = 35$ or 33; $m = n = 2 \cdot 3 \cdot 13$ gives $p = q = 59$, $r = 599$, $t = 59 \cdot 61$; $m = 3$, gives $p = 29$, $q = 839$, $r = 13 \cdot 17 \cdot 19$. For $f = 1, 7, 11$, or 25, there is no set m, n. For $f = 125$, $e = 60$, $M = 24m$, $mn = 2^2 \cdot 3^2 \cdot 11 \cdot 13 \cdot 127$, $p + 1 = 2(m + 936)/5$, $m \equiv 4 \pmod 5$, so that $m = 4, 9, 3 \cdot 13$ or $2 \cdot 3^2 \cdot 13$ or their products by 11. Then p is composite except for $m = 3 \cdot 13$, $q = 17 \cdot 727$, and for $m = 2 \cdot 3^2 \cdot 13$, $p = 467$, $q = 2371$, $r \equiv 0 \pmod 5$.

11. For $a = 192 = 2^6 \cdot 3$, $b = 48$, $c = 17$. If $f = 5$, $e = 54$, $M = 36m$, $mn = 4 \cdot 31$, $p + 1 = 2(m + 8)/3$; either $m = 1$, $q = 3 \cdot 29$ or $m = 4$, $q = 25$. If $f = 7$, $e = 8 \cdot 11$, $M = 16m$, $mn = 2^2 \cdot 3 \cdot 5^3$, $p + 1 = 2(m + 24)/11$, $m = 20$, $p = 7$, $q = 17$, $r = 17$, $t = 11 \cdot 13$. If $f = 35$, $e = 64 \cdot 3$, $M = 128 \cdot 3m$, $mn = 3 \cdot 47$, $p + 1 = 2m + 12$; if $m = 1$, $p = 13$, $q = 293$, but $r + 1$ is not an integer; if $m = 3$, $q = 105$. If $f = 55$, $e = 8 \cdot 51$, $M = 48m$, $mn = 2^2 \cdot 3^2 \cdot 13 \cdot 83$, $m = 13$ or $2 \cdot 3^2 \cdot 13$, $p = 9$ or 35. If $f = 1, 11, 13$, or 25, there is no set m, n.

12. For $a = 2^7 \cdot 3$, $b = 32$, $c = 11$. If $f = 5$, $e = 34$, $M = 4m$, $mn = 2^3 \cdot 3 \cdot 181$, $m = 3$, $q = 175$. If $f = 7$, $e = 8 \cdot 7$, $M = 16m$, $mn = 2^3 \cdot 3^4$, $m = 2^2 \cdot 3$, $p = 7$, $q = 19$, $r = 19$, $t = 3 \cdot 53$. If $f = 11$, $p = 3$, excluded. If $f = 35$, $e = 16 \cdot 7$, $M = 32m$, $mn = 2^3 \cdot 3 \cdot 11 \cdot 31$, $p + 1 = 2(m + 48)/7$; $m = 1$, $p = 13$, $q = 2351$, $r = 685$; $m = 8$, $p = 15$; $m = 22$, $q = 7 \cdot 17$. For $f = 1, 13, 55$ or 175, there is no set m, n.

13. For $a = 2^4 \cdot 3^2 \cdot 17$, $b = 8 \cdot 17$, $c = 5$. If $f = 1$, $e = 5$, $M = 2m$, $mn = 2 \cdot 3 \cdot 17 \cdot 47$, $m \equiv 2 \pmod 5$; $m = 2$, $p = 27$; $m = 17$, $p = 33$; $m = 6 \cdot 17$, $p = 67$, $q = 45$. If f is a prime, $e = 5f - 131$, $f > 26$. For $f = 29$, $e = 14$, $M = 4m$, $mn = 2 \cdot 3 \cdot 5 \cdot 17 \cdot 2243$, $m \equiv 2 \pmod 7$; $m = 2$, $p = 3 \cdot 97$; $m = 30$, $p = 13 \cdot 23$; $m = 3 \cdot 17$, $p = 305$; $m = 2 \cdot 5 \cdot 17$, $p = 3 \cdot 113$. For $f = 37$, 41, 43, or 29^2, there is no set m, n.

14. We may employ a simple tentative method of finding amicable triples ar, as, at, where a is relatively prime to r, s, t. Let r, s, t be chosen so that $\sigma(r) = \sigma(s) = s(t)$. The problem is to find a such that also

$$\sigma(r) \cdot \sigma(a) = (r + s + t)a.$$

The set r, s, t is to be rejected if $\sigma(r)$ has a prime factor dividing one of the numbers r, s, t, but not their sum, since it must then divide a. All such sets in which the common divisor sum $\sigma(r)$ does not exceed 360 are given in the following list

$$\begin{array}{rrrr}
2^2 \cdot 11, & 5 \cdot 13, & 83, & 7\sigma(a) = 2^4 a; \\
2 \cdot 5 \cdot 7, & 5 \cdot 23, & 7 \cdot 17, & 3^2\sigma(a) = 19a; \\
5 \cdot 31, & 7 \cdot 23, & 191, & 2^6\sigma(a) = 13^2 a; \\
7 \cdot 29, & 11 \cdot 19, & 239, & 2^4 \cdot 5\sigma(a) = 7 \cdot 31a; \\
2 \cdot 5 \cdot 13, & 5 \cdot 41, & 13 \cdot 17, & 3^2 \cdot 7\sigma(a) = 139a; \\
5 \cdot 41, & 13 \cdot 17, & 251, & 2^2 \cdot 3^2 \cdot 7\sigma(a) = 677a; \\
2 \cdot 83, & 5 \cdot 41, & 13 \cdot 17, & 3^2 \cdot 7\sigma(a) = 2^2 \cdot 37a; \\
3 \cdot 89, & 11 \cdot 29, & 17 \cdot 19, & 2^3 \cdot 5\sigma(a) = 101a; \\
11 \cdot 29, & 17 \cdot 19, & 359, & 2^3 \cdot 3^2 \cdot 5\sigma(a) = 7 \cdot 11 \cdot 13a; \\
3 \cdot 89, & 11 \cdot 29, & 359, & 2^3\sigma(a) = 3 \cdot 7a.
\end{array}$$

For the final equation the solution $a = 2^5 \cdot 3$ obtained at the end of § 8 is now excluded since 3 divides one of the triple. A valid solution may be found tentatively. Let $a = 2^n b$. For certain values of $n < 14$, a would necessarily be divisible by one of the prime factors of the numbers of the triple; for the others, a solution seems improbable. Take $n = 14$. Then $31 \cdot 151\sigma(b) = 2^{11} \cdot 3b$. Let $b = 31 \cdot 151c$. Then $19\sigma(c) = 2^3 \cdot 3c$. Let $c = 19d$. Then $5\sigma(d) = 2 \cdot 3d$, $d = 5$. Hence we have the amicable triple

$$\text{(X)} \qquad 3 \cdot 89a, \quad 11 \cdot 29a, \quad 359a, \quad a = 2^{14} \cdot 5 \cdot 19 \cdot 31 \cdot 151.$$

MATHEMATICAL LITERATURE FOR HIGH SCHOOL TEACHERS.[1]

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There is no sharp line of division between the mathematics for the high school teacher and that which is more advanced. All teachers should strive to increase their knowledge of the subjects which they teach. The teacher who does not hunger and thirst for more knowledge cannot hope to inspire his students with the proper zeal and earnestness. The desire for more knowledge should be based, in part, on the fact that we seek more light on the subjects which we teach.

Mathematics Not a Finished Science.—Even such elementary concepts as those of the natural numbers become more replete with meaning if we study the efforts to base these concepts on those of the theory of aggregates. Questions relating directly to the natural numbers have been considered in recent years by some of the foremost mathematicians, especially in view of the fact that the

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[1] Address before the mathematical section of the High School Conference held at the University of Illinois in November, 1912. Reported by Dr. W. W. Denton.